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Continued Fractions of Operator-Valued Analytic Functions*

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The subject of this work is the convergence of infinite continued fractions whose coefficients are analytic functions which take their values in the space of bounded linear operators on a complex Hilbert space. By appropriately combining a prior result of Fair with Vitali's theorem, we show that convergence under the uniform operator topology occurs on certain angular regions of the complex plane whenever the operator coefficients are commutative, invertible, and satisfy certain conditions on their numerical ranges.

W. Fair [2] has recently published a result concerning the convergence of an infinite continued fraction whose coefficients are positive invertible commuting bounded linear operators on a Hilbert space. On the other hand, electrical networks whose resistive, inductive, and capacitive elements are operators of this type do have physical significance [8]. Moreover, the driving-point impedance of any such infinite ladder network is formally an infinite continued fraction having a form similar to that considered by Fair but whose coefficients are analytic operator-valued functions on a half-plane. It is therefore of interest to establish conditions under which the latter infinite continued fraction converges. This is the objective of the present paper, and it is acheived by appropriately combining Fair's theorem with Vitali's theorem.

Actually, our analysis can be extended quite easily to continued fractions having a more general form than that allowed by electrical networks; this we shall do. The special cases that are of significance for ladder networks are obtained by setting $m = 1$ in Section 2 and $m = 1/2$ in Section 3. Moreover, we investigate three classes of continued fractions. The first of these is discussed in the next section and has as its prototype the continued fraction

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generated by a ladder network containing resistors, inductors, and capacitors. The other two are discussed in the last section and correspond to ladder networks with only resistors and inductors or, alternatively, with only resistors and capacitors.

In the following, C is the complex plane, H is a complex Hilbert space, and I is the identity operator on H. The inner product for H is denoted by $(\cdot, \cdot)$. By an "operator" we mean a bounded linear operator on H with range in H. $[H; H]$ denotes the space of all operators and is endowed with the uniform operator topology. We use the standard notation

$$S_n = \frac{1}{B_1} \mathbin{+} \frac{1}{B_2} \mathbin{+} \cdots \mathbin{+} \frac{1}{B_n} \tag{1}$$

for a finite continued fraction whose partial numerators are unity and whose partial denominators are B_k.

Essential to our discussion is the following.

FAIR'S THEOREM. *Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of positive commuting members of $[H; H]$ such that $B_k > \delta_k I$, where the δ_k are positive numbers and $\sum_{k=1}^{\infty} \delta_k = \infty$. Then, each $S_n \in [H; H]$, and $\{S_n\}_{n=1}^{\infty}$ converges under the uniform operator topology.*

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The coefficients of the continued fractions in which we are interested are members of certain classes of analytic functions, which we denote by $\mathscr{P}_m$ and $\mathscr{P}_{i,m}$ and define as follows. We let m be a real number such that $1/2 \leqslant m < \infty$ and set

$$C_m = \{\zeta \in C: |\arg \zeta| < \pi/2m\}.$$

(We are, of course, using the principal branch of the "arg" function.) We also specify that the origin does not belong to C_m. Thus, C_m is an open set. For any fixed $\zeta \in C_m$, we let

$$\Omega_{m,\zeta} = \{\lambda \in C: |\arg \lambda| \leqslant m |\arg \zeta|\}$$

and take the origin to be a member of $\Omega_{m,\zeta}$. Thus, $\Omega_{m,\zeta}$ is a closed cone whose sides make an angle that is less than π and has the real axis as its bisector. Finally, $W[B]$ denotes the numerical range of the operator B; that is,

$$W[B] = \{(Ba, a): a \in H \text{ and } \| a \| = 1\}.$$

$W[B]$ is a bounded set whenever B is a bounded operator.

DEFINITION 1. $\mathscr{P}_m$ is the set of all analytic operator-valued functions F on C_m such that, for each $\zeta \in C_m$, $W[F(\zeta)] \subset \Omega_{m,\zeta}$.

We mention in passing that $F \in \mathscr{P}_m$ if and only if the following three conditions hold: $F(\zeta^{1/m})$ is an analytic function of ζ on the domain C_1. For every $\zeta \in C_1$ and every $a \in H$, $\mathrm{Re}(F(\zeta^{1/m})a, a) \geqslant 0$; that is, $F(\zeta^{1/m})$ is accretive. For each $\sigma > 0$, $F(\sigma)$ is self-adjoint. The "only if" part of this assertion is clear, and the "if" part follows from a standard result concerning positive-real functions [6, Theorem 5].

DEFINITION 2. $\mathscr{P}_{i,m}$ is the set of all $F \in \mathscr{P}_m$ such that, for each $\zeta \in C_m$, $W[F(\zeta)]$ is bounded away from the origin.

If $F \in \mathscr{P}_{i,m}$, then $F(\zeta)$ is invertible for every $\zeta \in C_m$. Indeed, the spectrum $\Lambda[F(\zeta)]$ of $F(\zeta)$ is contained in the closure of $W[F(\zeta)]$ [4, p. 111], and therefore the origin is not in $\Lambda[F(\zeta)]$, which implies our assertion.

As an example of an $F \in \mathscr{P}_{i,m}$, we have the finite sum

$$F(\zeta) = \sum_{p=1}^{q} A_p \zeta^{r_p}, \tag{2}$$

where the exponents r_p are real numbers satisfying $-m \leqslant r_p \leqslant m$ and the coefficients A_p are positive operators with at least one of them being invertible. (Here again, when r_p is not an integer, the principal branch of ζ^{r_p} is understood.) Indeed, $W[F(\zeta)]$ is clearly contained in $\Omega_{m,\zeta}$ for each $\zeta \in C_m$. Moreover, the positivity and invertibility of one of the coefficients, for example, A_k, implies that $W[A_k]$ is real and $\inf W[A_k] > \delta$ for some real $\delta > 0$ because $\inf W[A_k]$ is in the spectrum of A_k. Consequently, for every $\zeta \in C_m$, $W[F(\zeta)]$ is bounded away from the origin by at least the distance $\delta \cos \theta$, where $\theta = m \mid \arg \zeta \mid < \pi/2$.

A similar argument establishes the following lemma.

LEMMA 1. *If $F \in \mathscr{P}_{i,m}$ and $G \in \mathscr{P}_m$, then $F + G \in \mathscr{P}_{i,m}$.*

LEMMA 2. *Let $F \in \mathscr{P}_{i,m}$ and let F^{-1} denote the function $\zeta \mapsto [F(\zeta)]^{-1}$. Then, $F^{-1} \in \mathscr{P}_{i,m}$.*

Proof. That F^{-1} is analytic at every point ζ where F is analytic and $F(\zeta)$ is invertible can be shown in the same way as in the scalar case. (See Ref. 7, Problem 8.15-1.)

Now, by the invertibility of $F(\zeta)$, given any $a \in H$ with $\| a \| = 1$ we can find a $b \in H$ such that $F(\zeta)b = a$. Therefore, $(F^{-1}(\zeta)a, a) = \overline{(F(\zeta)b, b)}$, where the bar denotes the complex conjugate. Moreover, $N \leqslant \| b \| \leqslant \| F^{-1}(\zeta) \|$, where $N > 0$ is a lower bound on $F^{-1}(\zeta)$. (See Ref. 1, pp. 155–156.) It follows that $W[F^{-1}(\zeta)]$ is contained in the union of all sets $\overline{\alpha W[F(\zeta)]}$, where α varies through the interval $N^2 \leqslant \alpha \leqslant \| F^{-1}(\zeta)\|^2$. But, $W[F(\zeta)]$ is a bounded subset of $\Omega_{m,\zeta}$ and is bounded away from the origin. In view of the fact that $\Omega_{m,\zeta} = \overline{\Omega_{m,\zeta}}$, we can conclude that $W[F^{-1}(\zeta)]$ has the same properties as $W[F(\zeta)]$. So, truly, $F^{-1} \in \mathscr{P}_{i,m}$.

The next theorem is the main conclusion of this work and is concerned with the functions Z_n: $C_m \to [H; H]$, where

$$Z_n(\zeta) = \frac{1}{F_1(\zeta)} + \frac{1}{F_2(\zeta)} + \cdots + \frac{1}{F_n(\zeta)}, \tag{3}$$

n is any positive integer, and the F_k are members of $\mathscr{P}_{i,m}$. We also consider the function Y_n: $C_m \to [H; H]$ defined by

$$Y_n(\zeta) = \frac{1}{F_2(\zeta)} + \frac{1}{F_3(\zeta)} + \cdots + \frac{1}{F_n(\zeta)}. \tag{4}$$

It follows immediately from Lemmas 1 and 2 that both Z_n and Y_n are members of $\mathscr{P}_{i,m}$.

The symbol J appearing in condition (iii) below denotes a subset of the real positive axis having a finite positive point of accumulation. In regard to condition (ii), for any set $X \subset C$, Re X denotes $\{\text{Re } x\colon x \in X\}$.

THEOREM 1. *Assume that the coefficients in* (3) *satisfy the following three conditions*:

(i) $F_k \in \mathscr{P}_{i,m}$ *for every* k.

(ii) *Given any compact set* $\Xi \subset C_m$, *there exists a constant* $\delta > 0$ *such that* $\inf \text{Re } W[F_1(\zeta)] > \delta$ *for all* $\zeta \in \Xi$.

(iii) *There exists a* J *such that, for each* $\sigma \in J$, *the* $F_k(\sigma)$ *commute with each other and* $F_k(\sigma) > \delta_k(\sigma)I$, *where the* $\delta_k(\sigma)$ *are positive numbers satisfying* $\sum_{k=1}^{\infty} \delta_k(\sigma) = \infty$.

Then, for every $\zeta \in C_m$, $\{Z_n(\zeta)\}_{n=1}^{\infty}$ *converges in the uniform operator topology, and the convergence is uniform with respect to* ζ *in any compact subset of* C_m.

Before proving this theorem, let us point out an example that satisfies the theorem's hypothesis. If every F_k has the form of (2) with the same conditions on r_p and A_p as before, then conditions (i) and (ii) are satisfied. If, in addition, every coefficient A_p commutes with all the other coefficients in all of the terms $F_k(\zeta)$, then the commutativity assertion in condition (iii) is fulfilled. Finally, a sufficient (but by no means necessary) condition for the satisfaction of the rest of condition (iii) is the following. For every k, there exists a term $A_{k,p}\zeta^{r_{k,p}}$ in the finite sum for $F_k(\zeta)$ such that $A_{k,p} > \delta_k I$, where the δ_k are positive numbers and $\sum_{k=1}^{\infty} \delta_k = \infty$. In this case, the set J can be taken to be the real positive axis.

When all $r_{k,p}$ are equal to either -1, 0, or 1, we have the form for the driving-point impedance of an infinite resistor–inductor–capacitor ladder network.

Proof of Theorem 1. If an invertible operator A is such that $\inf \operatorname{Re} W[A] > \delta > 0$, then $\| A^{-1} \| < \delta^{-1}$. Indeed, for every $a \in H$, $\delta \| a \|^2 < \operatorname{Re}(Aa, a) \leqslant \| Aa \| \| a \|$ so that $\delta \| a \| < \| Aa \|$ or equivalently $\| A^{-1} \| < \delta^{-1}$.

Now, let Ξ and δ be as in condition (ii). Since $Y_n \in P_{i,m}$, we have, for every $\zeta \in \Xi$ and for all n, that

$$\inf \operatorname{Re} W[F_1(\zeta) + Y_n(\zeta)] > \delta.$$

Upon identifying A with the invertible operator $[Z_n(\zeta)]^{-1} = F_1(\zeta) + Y_n(\zeta)$, we see that $\| Z_n(\zeta) \| < \delta^{-1}$ for every $\zeta \in \Xi$ and for all n.

On the other hand, for a fixed $\sigma \in J$, the $F_k(\sigma)$ are positive operators and, by condition (iii), satisfy the hypothesis of Fair's theorem. Hence, $\{Z_n(\sigma)\}_{n=1}^{\infty}$ converges in the uniform operator topology for each $\sigma \in J$. Since Ξ can be chosen as any compact subset of C_m, we may now invoke Vitali's theorem [5, p. 104] to conclude that $\{Z_n(\zeta)\}_{n=1}^{\infty}$ converges in the uniform operator topology for each $\zeta \in C_m$ and that the convergence is uniform on every compact subset Ξ of C_m.

COROLLARY 1a. *In addition to the hypothesis of Theorem* 1, *assume that* F_2 *also satisfies condition* (ii). *Let* Z *denote the limit of* $\{Z_n\}_{n=1}^{\infty}$. *Then*, $Z \in \mathscr{P}_{i,m}$.

Proof. As $n \to \infty$, $(Z_n(\zeta)a, a) \to (Z(\zeta)a, a)$ for every $\zeta \in C_m$ and every $a \in H$. But, $(Z_n(\zeta)a, a) \in \Omega_{m,\zeta}$, and therefore so too is $(Z(\zeta)a, a)$. In other words, $Z \in \mathscr{P}_m$. By the same considerations, the limit Y of $\{Y_n\}_{n=1}^{\infty}$ exists and is a member of $\mathscr{P}_m$. Since $F_1 \in \mathscr{P}_{i,m}$, Lemmas 1 and 2 indicate that $F_1 + Y \in \mathscr{P}_{i,m}$ and $Z = (F_1 + Y)^{-1} \in \mathscr{P}_{i,m}$.

Before leaving this section, we mention that, if condition (iii) is replaced by the weaker assumption that all the $F_k(\sigma)$ commute for each $\sigma \in J$, it can still be shown through much the same kind of reasoning that $\{Z_{2n-1}(\zeta)\}_{n=1}^{\infty}$ and $\{Z_{2n}(\zeta)\}_{n=1}^{\infty}$ converge in the strong operator topology uniformly on every compact subset of C_m. Moreover, the limits of both sequences are members of $\mathscr{P}_{i,m}$.

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We now take up the continued fractions whose prototypes are the driving-point impedances of resistor–inductor or resistor–capacitor ladder networks.

We again let $1/2 \leqslant m < \infty$ and define C_m as before. For any fixed $\zeta \in C_m$, we define the closed set $\Theta_{m,\zeta}$ as follows.

$$\Theta_{m,\zeta} = \{\lambda \in C: 0 \leqslant \arg \lambda \leqslant 2m \arg \zeta\} \quad \text{when} \quad \operatorname{Im} \zeta \geqslant 0;$$

$$\Theta_{m,\zeta} = \{\lambda \in C: 2m \arg \zeta \leqslant \arg \lambda \leqslant 0\} \quad \text{when} \quad \operatorname{Im} \zeta \leqslant 0;$$

The origin is taken to be a member of $\Theta_{m,\zeta}$.

DEFINITION 3. $\mathscr{M}_{1,m}$ is the set of all analytic operator-valued functions F on C_m such that $W[F(\zeta)] \subset \Theta_{m,\zeta}$ for every $\zeta \in C_m$. $\mathscr{M}_{1i,m}$ is the set of $F \in \mathscr{M}_{1,m}$ such that, for each $\zeta \in C_m$, $W[F(\zeta)]$ is bounded away from the origin. Similarly, $\mathscr{M}_{2,m}$ is the set of all analytic operator-valued functions F on C_m such that $W[F(\zeta)] \subset \Theta_{m,\bar{\zeta}}$ for every $\zeta \in C_m$. ($\bar{\zeta}$ denotes the complex conjugate of ζ.) $\mathscr{M}_{2i,m}$ is the set of all $F \in \mathscr{M}_{2,m}$ such that, for each $\zeta \in C_m$, $W[F(\zeta)]$ is bounded away from the origin.

In much the same way as Lemmas 1 and 2 are established, we can prove the following.

LEMMA 3. *If $F \in \mathscr{M}_{1i,m}$ and $G \in \mathscr{M}_{1,m}$, then $F + G \in \mathscr{M}_{1i,m}$. If $F \in \mathscr{M}_{2i,m}$ and $G \in \mathscr{M}_{2,m}$, then $F + G \in \mathscr{M}_{2i,m}$.*

LEMMA 4. *Let F^{-1} denote the function $\zeta \mapsto [F(\zeta)]^{-1}$. If $F \in \mathscr{M}_{1i,m}$, then $F^{-1} \in \mathscr{M}_{2i,m}$. If $F \in \mathscr{M}_{2i,m}$, then $F^{-1} \in \mathscr{M}_{1i,m}$.*

In the following we again define Z_n, Y_n, and J as before. $P_{m,\zeta}$ denotes the projection of C onto the bisector of the angular region $\Theta_{m,\zeta}$. Also, for any set $X \subset C$, $|X|$ denotes $\{|x| : x \in X\}$.

THEOREM 2. *Assume that the coefficients in* (3) *and* (4) *satisfy the following three conditions*:

(i) *$F_k \in \mathscr{M}_{1i,m}$ for k odd, and $F_k \in \mathscr{M}_{2i,m}$ for k even.*

(ii) *Given any compact set $\Xi \subset C_m$, there exists a constant $\delta > 0$ such that $\inf |P_{m,\zeta} W[F_1(\zeta)]| > \delta$ for all $\zeta \in \Xi$.*

(iii) *There exists a J such that, for each $\sigma \in J$, the $F_k(\sigma)$ commute with each other and $F_k(\sigma) > \delta_k(\sigma) I$, where the $\delta_k(\sigma)$ are positive numbers and $\sum_{k=2}^{\infty} \delta_k(\sigma) = \infty$.*

Then, for each $\zeta \in C_m$, $\{Z_n(\zeta)\}_{n=1}^{\infty}$ converges in the uniform operator topology, and the convergence is uniform with respect to ζ in any compact subset of C_m.

Proof. By Lemmas 3 and 4, $Z_n \in \mathscr{M}_{2i,m}$ and $Y_n \in \mathscr{M}_{1i,m}$. We now proceed as in the proof of Theorem 1, but fix our attention upon the angular region $\Theta_{m,\zeta}$ rather than upon $\Omega_{m,\zeta}$. By condition (ii) and the fact that $Y_n \in \mathscr{M}_{1i,m}$,

$$\inf | P_{m,\zeta} W[F_1(\zeta) + Y_n(\zeta)]| > \delta,$$

where δ does not depend on the choice of $\zeta \in \Xi$ or of n. This implies that $\| Z_n(\zeta)\| < \delta^{-1}$ for every $\zeta \in \Xi$ and every n. The proof is completed by using Fair's theorem and Vitali's theorem as before.

Similarly, the proof of Corollary 1a can be modified to yield the next corollary.

COROLLARY 2a. *In addition to the hypothesis of Theorem 2, assume that F_2 also satisfies condition* (ii) *of Theorem 2. Let $Z = \lim Z_n$. Then, $Z \in \mathscr{M}_{2i,m}$.*

The following result can also be obtained without much change in the arguments of this section.

THEOREM 3. *Let the hypothesis of Theorem 2 hold, but with the modification that F_2 replaces F_1 in condition* (ii). *Then, the conclusion of Theorem 2 holds for $\{Y_n(\zeta)\}_{n=1}^{\infty}$. If, in addition, F_3 also satisfies condition* (ii) *of Theorem 2 and if $Y = \lim Y_n$, then $Y \in \mathscr{M}_{1i,m}$.*

An example of a continued fraction that satisfies the hypotheses of Theorems 2 and 3 and Corollary 2a is the one whose coefficients are given by

$$F_k(\zeta) = \sum_{p=1}^{q_k} A_{k,p} \zeta^{r_{k,p}}, \qquad k = 1, 2, \ldots$$

and satisfy the following conditions. The $A_{k,p}$ are positive operators and commute with all other $A_{i,j}$. For k odd, $0 \leqslant r_{k,p} \leqslant 2m$. For k even, $-2m \leqslant r_{k,p} \leqslant 0$. Finally, for every k, there exists an operator $A_{k,p} > \delta_k I$, where the δ_k are positive numbers satisfying $\sum \delta_k = \infty$.

When all $r_{k,p}$ are equal to either 0 or 1 for k odd and to either 0 or -1 for k even, Z (or Y) has the form for the driving-point impedance of a resistor–capacitor (or respectively, resistor–inductor) infinite ladder network.

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